Effect of α quenching on magnetic field size

Manuel Núñez
Departamento de Análisis Matemático, Universidad de Valladolid, 47005 Valladolid, Spain
(Received 12 December 2000; published 17 April 2001)

It is commonly assumed that the α effect of mean-field magnetohydrodynamics essentially stops acting wherever the mean-field size reaches a certain value. We show that if the mean velocity is approximately constant, the regions where the field reaches such a threshold tend to shrink in size or the field tends to become constant there. The rate of this process is also estimated.

DOI: 10.1103/PhysRevE.63.056404 PACS number(s): 52.35.Ra, 52.30.Cv, 95.30.Qd

I. INTRODUCTION

Under the magnetohydrodynamic approximation, the magnetic field **B** in a plasma of velocity **u** and resistivity η satisfies the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \Delta \mathbf{B} + \nabla \times (\mathbf{u} \times \mathbf{B}). \tag{1}$$

This equation, although linear in **B**, becomes useless for computational purposes if the velocity **u** is turbulent. However, the presence of large-scale features of the magnetic field has been observed even in this situation. Apparently there is an inverse cascade from small to large scales for the field. While several modelizations seem to indicate that this cascade proceeds in a classical way, transferring gradually magnetic energy to ever larger scales (see e.g., Ref. [1]), other authors rather think that there is direct production of large-scale fields from small-scale ones [2]. This, however, will not affect our analysis. The correct procedure to study this phenomenon should be integration of the full magnetohydrodynamics (MHD) system. Unfortunately, this is computationally difficult and analytically almost impossible except in simple cases. Under certain hypotheses, however, a simpler model has been proposed involving only the large-scale components of the magnetic field and the velocity (also denoted by **B** and **u**). They are supposed to satisfy the equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [-(\eta + \beta) \nabla \times \mathbf{B} + \mathbf{u} \times \mathbf{B} + \alpha \mathbf{B}]. \tag{2}$$

where β is a turbulent diffusivity and α is the factor representing the enhancement of the magnetic field by small-scale turbulent velocity. α is a scalar only in the case of isotropic turbulence. As a matter of fact there should be some kind of projection of the right-hand side term into some space of large-scale fields to make Eq. (2) a closed equation. For periodic problems, this space may be identified with the set of functions whose Fourier modes do not exceed a certain value; in other cases the relevant space is left rather loose. The space generated by the first few eigenfunctions of the Laplacian with the right boundary conditions is a good candidate.

There exists a classical derivation of Eq. (2), the so-called equation of mean-field magnetohydrodynamics, for small fields (see e.g. Ref. [3]). Although many researchers find it convincing, let us just say that it is not rigorous [4,5]. It is probably safer to view Eq. (2) as a semiempirical formula, which has proved indeed very successful for modelization of many astrophysical phenomena (see e.g. Ref. [2] and references therein). Nothing but qualitative predictions should be expected from it, however.

Although, as asserted, the regular procedure should be to solve the MHD system, by considering the effect upon the velocity of the Lorentz force, this is so costly that it is natural to try to extend Eq. (2) beyond its original range of weak fields. However, it is known that large fields will tend to suppress turbulence and therefore both α and β should decrease with field size. The precise dependence, being to some extent a matter of convenience to meet experimental data, is more polemic. The standard one for α is

$$\alpha = \frac{f}{1 + kB^2}, \tag{3}$$

for some bounded function f and positive constant k. For axisymmetric systems, where Eq. (2) is most often applied, f is usually taken as a multiple of $\cos\theta$, where θ is the latitude coordinate. There is some controversy on the size of k. It has been argued [6,7] that k could be of the order of $1/\eta$, a large amount because the resistivity of astrophysical plasmas is usually very low. If so, the threshold beyond which the α term is insignificant, could occur quite soon.

Our purpose is to study the effect of this threshold upon the magnetic field size. Since we will try to isolate the contribution of α, we will take the mean velocity as constant, so that it may be eliminated by a Galilean transformation. Also for simplicity purposes we will take the resistivity as constant, ignoring the turbulent contribution, since any additional resistivity will only tend to smooth the field, our results will remain valid (and even stronger) in the general case. We will make no hypotheses upon the specific form of the function α, only we assume that beyond a certain value B_0 of the field, it is small enough to be safely ignored. Under these conditions, we will prove that eventually the field will not exceed B_0. Moreover, if some region originally exceeds this value, it will tend to form a plateau of constant field or the region itself shrinks in volume, and the sharper the original field gradients are, the more rapid is this process. Let us note that these results do not follow in any obvious way from

Eq. (2): although the equation is dissipative for $\alpha=0$, there could exist regions in the plasma where $B<B_0$ and therefore α is positive; these regions may vary in time, and their effects upon the whole domain must be dealt with.

These results could conceivably throw light upon the previously mentioned controversy upon the size of k, since for larger k, the cutoff B_0 is smaller. In the case of early alpha quenching, therefore, not only the magnetic energy will tend to be small in time, but the size of the field itself is uniformly bounded by a small constant for large times.

II. THE MAIN ESTIMATES

Let $\phi:[0,\infty)\to[0,\infty)$ be a positive, increasing and twice differentiable function. Since we assume that the threshold of α depends only on B^2, we will consider the function $\phi(B^2)$. By elementary operations,

$$\frac{\partial}{\partial t}(\phi\circ B^2)=2\phi'(B^2)\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t},$$

$$\Delta(\phi\circ B^2)=2\phi'(B^2)|\nabla\mathbf{B}|^2+\phi''(B^2)|\nabla B^2|^2+2\phi'(B^2)\mathbf{B}\cdot\Delta\mathbf{B}.$$

Since we assume that the mean velocity is zero, the function $\phi\circ B^2$ satisfies the equation

$$\frac{\partial}{\partial t}(\phi\circ B^2)=\eta\Delta(\phi\circ B^2)-\eta\phi''(B^2)|\nabla B^2|^2-2\eta\phi'(B^2)|\nabla\mathbf{B}|^2+2\phi'(B^2)\mathbf{B}\cdot[\nabla\times(\alpha\mathbf{B})]. \tag{4}$$

Let us integrate all the terms in the domain Ω under consideration, which we assume smooth enough. Obviously

$$\int_\Omega\Delta(\phi\circ B^2)dV=\int_{\partial\Omega}\frac{\partial(\phi\circ B^2)}{\partial n}d\sigma.$$

This boundary integral vanishes with periodic, Dirichlet, or Neumann homogeneous conditions, or in the absence of boundary. More generally, it is negative as long as $\phi\circ B^2$ decreases towards the boundary of Ω. Thus, for instance, since $\phi'\geqslant0$, if $\partial B^2/\partial n\leqslant0$, the integral is negative. This often happens in real situations: one chooses the domain so that the magnetic field is concentrated there and it decreases in size towards $\partial\Omega$. From now on we will assume that this integral is not positive.

Finally, since

$$2\phi'(B^2)\mathbf{B}\cdot[\nabla\times(\alpha\mathbf{B})]=2\phi'(B^2)\alpha\mathbf{B}\cdot(\nabla\times\mathbf{B})=2\phi'(B^2)\alpha\mathbf{B}\cdot\mathbf{J},$$

where $\mathbf{J}$ is the plasma current, we are left with

$$\frac{\partial}{\partial t}\int_\Omega\phi(B^2)dV\leqslant-\eta\left(\int_\Omega\phi''(B^2)|\nabla B^2|^2+2\phi'(B^2)|\nabla\mathbf{B}|^2\,dV\right)+2\int_\Omega\phi'(B^2)\alpha\mathbf{B}\cdot\mathbf{J}\,dV. \tag{5}$$

Notice that since all the functions are large scale, the current cannot be too large if $\mathbf{B}$ is not, so that the contribution of small regions of Ω to the last term of Eq. (5) is equally small. Thus, if B_0 is the threshold beyond which α is small enough to be discounted and $\phi'(B^2)$ vanishes for B in $[0,B_0]$, the last integral may be considered as zero. For such functions ϕ,

$$\frac{\partial}{\partial t}\int_\Omega\phi(B^2)dV\leqslant-\eta\left(\int_\Omega\phi''(B^2)|\nabla B^2|^2+2\phi'(B^2)|\nabla\mathbf{B}|^2dV\right), \tag{6}$$

which is the main bound. It tells us that, provided $\phi',\phi''\geqslant0$, the integral of $\phi(B^2)$ decreases in time.

Every election of ϕ provides some insight on the behavior of B at the subdomain $B>B_0$. We will consider only the following examples:

$$\phi_1(x)=B_0^2,x\in[0,B_0];\quad \phi_1(x)=x^2,x>B_0,$$

$$\phi_2(x)=B_0,x\in[0,B_0];\quad \phi_1(x)=x,x>B_0.$$

In fact the function ϕ_2 is not twice differentiable at the point B_0, but by approximating it with smooth functions we will see that the integrals converge to appropriate limits. Now the inequality (6) becomes for ϕ_1

$$\frac{\partial}{\partial t}\int_{B>B_0}B^4-B_0^4dV\leqslant-\eta\left(\int_{B>B_0}2|\nabla B^2|^2+2B^2|\nabla\mathbf{B}|^2dV\right)\leqslant-4\eta B_0^2\int_{B>B_0}|\nabla\mathbf{B}|^2dV. \tag{7}$$

Hence the integral of $B^4-B_0^4$ decreases and is positive. Therefore it must tend to a constant and its time derivative to zero. For this to happen, $\int_{B>B_0}|\nabla\mathbf{B}|^2dV$ must tend to zero. This seems to indicate that $\mathbf{B}$ must tend to a constant at $B>B_0$, since the value of B at the boundary of this region is B_0, this constant value has size B_0. Actually this argument, if made in a more general case, would need some fine points of functional analysis, but since we are working within a finite-dimensional space of functions all the norms are equivalent and $\mathbf{B}$ tends uniformly to a constant field of magnitude B_0 in $B>B_0$, or perhaps this region tends to disappear. Anyway, we see that the magnetic field cannot exceed B_0 in the long run.

We also see that the larger $|\nabla\mathbf{B}|$, the faster the convergence. Let us see now how the gradient at the very edge B

$=B_0$ affects this convergence rate by analyzing ϕ_2. Since ϕ_2 and ϕ_2' are bounded functions, by approximating this function uniformly with smooth ones, it is straightforward to see that the terms

$$\frac{\partial}{\partial t}\int_\Omega \phi_2(B^2)dV,$$

$$\int_\Omega \phi_2'(B^2)|\nabla \mathbf{B}|^2 dV$$

correspond, respectively, to

$$\frac{\partial}{\partial t}\int_{B>B_0} B^2-B_0^2 dV$$

$$\int_{B>B_0}|\boldsymbol{\nabla}\mathbf{B}|^2 dV.$$

However, the term

$$\int_\Omega \phi_2''(B^2)|\boldsymbol{\nabla}B^2|^2 dV$$

needs a separate study. Let us recall that for any smooth real function v defined in Ω, the level sets $S_c: v=c$ are smooth surfaces for almost every real c, and for any continuous G

$$\int_\Omega G|\boldsymbol{\nabla}v|dV=\int_{-\infty}^{\infty}dv\int_{S_v}G\,d\sigma, \tag{8}$$

where $d\sigma$ denotes the surface area element (see e.g. Ref. [8]). Notice also that for a particular v the set S_v may not be a surface at all, and even to fill an open subset of Ω, but this only happens for a measure zero set of v's. Now, B is smooth as it lies in a space of smooth functions (the large-scale ones). By taking $v=B^2$, $G=\phi''$ for a smooth ϕ in Eq. (8),

$$\int_\Omega \phi''(B^2)|\boldsymbol{\nabla}B^2|^2 dV=\int_0^\infty \phi''(B^2)dB^2\int_{S_{B^2}}|\boldsymbol{\nabla}B^2|d\sigma, \tag{9}$$

since ϕ'' depends only on B^2. The second derivative of ϕ_2 in the sense of distributions is the Dirac measure $\delta_{B_0^2}$. Assuming B_0 is one of the (almost all) levels where $B=B_0$ is a smooth surface, the limit, when approximating ϕ_2 by smooth functions, of Eq. (8) is

$$\int_{B=B_0}|\boldsymbol{\nabla}B^2|d\sigma,$$

and therefore Eq. (6) becomes

$$\frac{\partial}{\partial t}\int_{B>B_0}B^2-B_0^2 dV\leqslant-\eta\left(\int_{B=B_0}|\boldsymbol{\nabla}B^2|d\sigma + \int_{B>B_0}|\boldsymbol{\nabla}\mathbf{B}|^2 dV\right). \tag{10}$$

The same conclusion as before may be reached now, but in addition we observe that the gradient of B^2 must be small at every level surface (since we may apply the previous argument to any $B_1>B_0$) or the convergence is faster. Thus, the sharpest the gradient of B^2 at $B=B_0$, the quickest **B** tends to flatten or the region to disappear.

III. CONCLUSIONS

We have analyzed the effect upon the magnetic field of the existence of a threshold B_0 beyond which the alpha term of mean-field magnetohydrodynamics becomes irrelevant. If we exclude the action of the mean velocity, we have shown that the region where the field size exceeds this threshold shrinks in volume and the field tends to become constant at the level $B=B_0$. The process is faster if originally there are large gradients of field at $B>B_0$, or gradients of field size at $B=B_0$. As a consequence, a rapid cutoff of the alpha factor will imply a smaller uniform bound upon the size of the magnetic field.

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